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Indagationes Mathematicae 23 (2012) 547–555

indagationes mathematicae

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On the law of large numbers and arithmetic functions

István Berkes[a,*], Wolfgang Müller[a,1], Michel Weber[b,2]

[a] *Institute of Statistics, Graz University of Technology, Münzgrabenstrasse 11, 8010 Graz, Austria*

[b] *IRMA, Institut de Recherche Mathématique Avancée, 10 rue du Général Zimmer, 67084 Strasbourg Cedex, France*

Received 22 October 2011; received in revised form 6 March 2012; accepted 17 April 2012

Communicated by R. Tijdeman

Abstract

In this paper we give an optimal condition for the strong law of large numbers

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} f(n)X_n}{\sum_{n=1}^{N} f(n)} \overset{\text{a.s.}}{=} \mathbf{E}\, X$$

where $X, X_1, X_2, \ldots$ are i.i.d. integrable random variables and f is an additive arithmetic function. We also give sufficient criteria for multiplicative weight functions f.

Keywords: Strong law of large numbers; Weighted sums of i.i.d. random variables; Additive and multiplicative functions

1. Introduction

Let $f \geq 0$ be an additive arithmetic function and put

$$A_n = \sum_{p^\nu \leq n} \frac{f(p^\nu)}{p^\nu}\left(1 - \frac{1}{p}\right), \qquad B_n = \sum_{p^\nu \leq n} \frac{|f(p^\nu)|^2}{p^\nu}. \tag{1}$$

* Corresponding author. Tel.: +43 316 873 6976; fax: +43 316 873 6977.
E-mail address: berkes@tugraz.at (I. Berkes).

[1] Tel.: +43 316 873 6479; fax: +43 316 873 6977.

[2] Tel.: +33 3 68 85 01 29; fax: +33 3 68 85 03 28.

doi:10.1016/j.indag.2012.04.002

Berkes and Weber [3] proved, under the additional assumption of strong additivity of f, that if

$$B_p \to \infty, \quad f(p) = o(B_p^{1/2}) \quad \text{as } p \to \infty, \tag{2}$$

then the weighted SLLN with coefficients $f(n)$ holds, i.e. for any i.i.d. sequence $X, X_1, X_2, \ldots$ with finite mean we have

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} f(n) X_n}{\sum_{n=1}^{N} f(n)} = \mathbf{E}\, X \quad \text{a.s.} \tag{3}$$

Fukuyama and Komatsu [5] weakened condition (2) (still in the strongly additive case) to the Lindeberg condition

$$\lim_{n\to\infty} \frac{1}{B_n} \sum_{\substack{p^\nu \le n \\ |f(p^\nu)| \ge \varepsilon B_n^{1/2}}} \frac{|f(p^\nu)|^2}{p^\nu} = 0 \quad \text{for any } \varepsilon > 0, \tag{4}$$

and Tenenbaum [9] weakened (4) further to

$$B_n = o(A_n^2), \tag{5}$$

assuming only the simple additivity of f. He also showed that

$$\sum_{p^\nu \le N} \frac{f(p^\nu)^3}{p^\nu} \ll A_N^3 \tag{6}$$

is a sufficient condition for the corresponding law of the iterated logarithm. In this paper we relax (5) further and give an optimal condition for the SLLN with additive weights. To simplify the formulations, let us say that the weighted SLLN holds if relation (3) is valid for all i.i.d. sequences $X, X_1, X_2, \ldots$ with finite mean. Also, call a nonnegative sequence (a_n) quasi-monotone if there exists a nonnegative monotone sequence (b_n) with $b_n \ll a_n \ll b_n$.

Theorem 1. *Let f be a complex valued additive function and define A_n and B_n by (1). Assume that*

$$n|A_n| \quad \textit{is quasi-monotone and } n|A_n| \to \infty \tag{7}$$

and

$$B_n \ll |A_n|^2. \tag{8}$$

Then the weighted SLLN holds. On the other hand, for every $\omega_n \to \infty$ there exists a strongly additive function f satisfying (7) and $B_n \sim \omega_n |A_n|^2$ such that the weighted SLLN fails.

The assumption of quasi-monotonicity in Theorem 1 is automatically satisfied if $f \ge 0$.

The proof of Theorem 1 will show that if $|f(p)| = O(1)$, then the relation $B_n/|A_n|^2 \to \infty$ implies the failure of the weighted SLLN. Thus for bounded $f(p)$ relation (8) is not only optimal, but it is close to a necessary and sufficient condition for the weighted SLLN. A simple calculation shows that assumptions (7) and (8) and thus the weighted SLLN are valid for $f(p) = (\log p)^\alpha$ for any $\alpha > 0$; on the other hand, it is not hard to show that for any $\omega_p \to \infty$ there exists a positive, strongly additive function f such that $f(p) \ll (\log p)^{\omega_p}$ and the weighted SLLN fails. It should be noted, however, that the weighted SLLN can hold also for some larger functions $f(p)$; for example, using the classical criterion of Jamison et al. [6] one can show that this is the case if $f(p) = p$.

We turn now to the case of multiplicative weight functions f, i.e. functions satisfying $f(mn) = f(m)f(n)$ for $(m, n) = 1$.

Theorem 2. *Let $f \geq 0$ be a multiplicative function. Assume that there are positive constants C_1, C_2, C_3, C_4 and $a > 1$ such that*

$$\sum_{p \leq x} f(p)^a \log p \leq C_1 x, \tag{9}$$

$$\sum_{p, \nu \geq 2} f(p^\nu)^a p^{-\nu} \log p^\nu \leq C_2 \tag{10}$$

$$\sum_{p \leq x} f(p) \log p \geq C_3 x \quad \textit{for } x \geq C_4. \tag{11}$$

Then the weighted SLLN is true.

As pointed out in [9], for any arithmetic function $f(n)$ and i.i.d. sequence (X_n) with finite variances, the validity of the law of the iterated logarithm for the sums $\sum_{n=1}^{N} f(n)X_n$ follows from the validity of the SLLN for X_n^2 with weights $f(n)^2$. Since for a multiplicative function $f(n)$ the squared function $f^2(n)$ is also multiplicative, it follows that the conditions of Theorem 2 with f replaced by f^2 imply the corresponding LIL. Since the square of an additive function is generally not additive, this argument does not work for additive functions and indeed, the LIL condition (6) of Tenenbaum is of a different character.

2. Examples

Our first example illustrates the big difference between the case $f \geq 0$ and the case of general real or complex valued f in Theorem 1. Let $f(n)$ be strongly additive such that $f(p)$ is constant on every residue class modulo some given $q \geq 1$, i.e.

$$f(p) = c(a) \quad \text{if} \quad p \equiv a \bmod q \qquad (a, q) = 1,$$

where $c(a)$ are arbitrary complex constants. This does not restrict $f(p)$ for the finitely many prime divisors p of q. Using the well known fact

$$\sum_{\substack{p \leq x \\ p \equiv a \bmod q}} \frac{1}{p} = \frac{1}{\varphi(q)} \log \log x + O(1) \qquad (a, q) = 1,$$

we obtain

$$A_n = \sum_{p \leq n} \frac{f(p)}{p} = \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} c(a) \sum_{\substack{p \leq n \\ p \equiv a \bmod q}} \frac{1}{p} = D_f \log \log n + O(1),$$

where

$$D_f = \frac{1}{\varphi(q)} \sum_{\substack{1 \leq a \leq q \\ (a,q)=1}} c(a).$$

Furthermore,

$$B_n \sim D'_f \log\log n \quad \text{with} \quad D'_f = \varphi(q)^{-1} \sum_{\substack{1\le a\le q\\(a,q)=1}} |c(a)|^2 > 0.$$

Thus Theorem 1 is applicable if $D_f \neq 0$ and the SLLN with weights $f(n)$ is true in this case. On the other hand, if $D_f = 0$, then $A_n \ll 1$ and $B_n \sim D'_f \log\log n$. Hence condition (8) is violated and the first part of Theorem 1 is not applicable. However, since in this case $|f(p)| = O(1)$ and $B_n/|A_n|^2 \to \infty$, the proof of the second statement of Theorem 1 in Section 3 shows that the SLLN with weights $f(n)$ is false. For instance, denote by $\omega_{a,q}(n)$ the number of prime divisors of n which are congruent to a modulo q, then the SLLN with weights $f(n) = \omega_{1,3}(n) - \omega_{2,3}(n)$ fails to be true, while the SLLN with weights $f(n) = \omega_{1,5}(n) + \omega_{2,5}(n) - \omega_{3,5}(n)$ is true.

We turn now to multiplicative examples.

(a) Let $f(n) = d(n^l)^\alpha$, with an integer $l \ge 1$ and real $\alpha > 0$. Then $f(p^\nu) = (\nu l + 1)^\alpha$. Similarly, if $f(n) = d_l(n)^\alpha$, where $d_l(n)$ denotes the number of ways to write n as a product of l integers, then $f(p^\nu) = \binom{l}{\nu}^\alpha$.
(b) Denote by $r(n)$ the number of ways to write $n = x^2 + y^2$ as a sum of two integer squares. Then $f(n) = \frac{1}{4} r(n)$ is multiplicative. More specifically $f(n) = \sum_{d|n} \chi_4(d)$, where χ_4 denotes the non trivial Dirichlet character modulo 4. Hence $f(p) = 2$ if $p \equiv 1 \bmod 4$ and $f(p) = 0$ if $p \equiv 3 \bmod 4$. More generally, let K be a quadratic number field and denote by $f(n)$ the norm counting function of K, i.e. $f(n)$ counts the number of ways to write n as the norm of an (integral) ideal. Then $f(n) = \sum_{d|n} \chi_D(n)$, where D denotes the discriminant of K and χ_D is the Dirichlet character modulo $|D|$ which is determined by the quadratic residue symbol $\chi_D(p) = \left(\frac{D}{p}\right)$, $p \nmid D$.
(c) Let $f(n)$ be the indicator function of all integers which are the sum of two squares, thus $f(n) = 1$ if $r(n) > 0$ and $f(n) = 0$ else. Then $f(n)$ is multiplicative with $f(p) = 1$ if $p \equiv 1 \bmod 4$ and zero else.

3. Proofs

Put

$$F_f(n) = \sum_{m\le n} f(m), \qquad G_f(n) = \sum_{m\le n} |f(m)|^2$$

and

$$N_f(x) = \#\{n : x|f(n)| \ge |F_f(n)|\}.$$

The following lemma is the complex version of the theorem of Jamison et al. [6]; for the proof see [2].

Lemma 3. *Assume that*

$$|F_f(n)| \quad \text{is quasi-monotone and } |F_f(n)| \to \infty. \tag{12}$$

Then the condition

$$N_f(x) \ll x \tag{13}$$

is necessary and sufficient for the SLLN with weights $f(n)$. Regardless of the validity of (12), condition (13) is necessary for the weighted SLLN.

Proof of Theorem 1. Clearly

$$\begin{aligned} F_f(n) &= \sum_{m\le n}\sum_{p^\nu\|m} f(p^\nu) = \sum_{p^\nu\le n} f(p^\nu)(\lfloor n/p^\nu\rfloor - \lfloor n/p^{\nu+1}\rfloor) \\ &= nA_n + O\left(\sum_{p^\nu\le n}|f(p^\nu)|\right) \\ &= nA_n + O(n(\log n)^{-1/2}B_n^{1/2}), \end{aligned} \tag{14}$$

by using the prime number theorem and the Cauchy–Schwarz inequality. Since by (8) the last remainder term in (14) is of smaller order than nA_n, it follows that $F_f(n)\sim nA_n$. On the other hand, the Turán–Kubilius inequality yields

$$\sum_{m\le n}|f(m)-A_n|^2 \ll nB_n,$$

whence we get, using $F_f(n)\sim nA_n$ and (8),

$$G_f(n) \ll n|A_n|^2 + nB_n \ll n|A_n|^2.$$

The so obtained relations are analogues of relations (6), (7) in [9] in the case when only (8) of the present paper is assumed; note that relation (8) of [9] also remains valid under condition (8) of our paper. Using these relations, the proof of Théorème 1 in [9] with minor modifications yields the weighted SLLN.

To prove the second part of Theorem 1, we first observe that if f satisfies the Lindeberg condition (4) and $B_n/|A_n|^2 \to \infty$, then the SLLN with weights $f(n)$ is false. Indeed, the Erdős–Kac–Kubilius CLT (see [7]), valid under the Lindeberg condition (4) and $|A_n| = o(B_n^{1/2})$ implies that $|f(n)| \gg B_n^{1/2}$ on a set H with positive density. From (4), (14) and $|A_n| = o(B_n^{1/2})$ we conclude $|F(n)/f(n)| = o(n)$ on H as $n\to\infty$, which implies that the necessary condition (13) of Lemma 3 is not satisfied and the weighted SLLN fails to be true. Now let $\omega_n \uparrow \infty$. It is easy to construct a set $H\subset\mathbb{N}$ such that

$$\sum_{p\le n,\, p\in H}\frac{1}{p} = \frac{1}{2}\log\log n + \frac{(1+o(1))}{2\sqrt{\omega_n}}\sqrt{\log\log n}.$$

Define a strongly additive function f by $f(p) = 1$ for $p\in H$ and $f(p) = -1$ for $p\notin H$. Clearly $B_n\sim \log\log n$ and thus the Lindeberg condition (4) holds, further

$$A_n = \sum_{p\le n,\, p\in H}\frac{1}{p} - \sum_{p\le n,\, p\notin H}\frac{1}{p} \sim \frac{1}{\sqrt{\omega_n}}\sqrt{\log\log n} \quad \text{as } n\to\infty$$

so that $B_n/|A_n|^2\sim\omega_n$, completing the proof. □

Proof of Theorem 2. Let f be a non-negative multiplicative function which satisfies (9)–(11) with some $a\ge 1$. Under stronger assumptions on f, Wirsing's theorem [10] gives the asymptotics of $F_f(x)$. We only use the estimate

$$F_f(x) \asymp \frac{x}{\log x}W_f(x), \tag{15}$$

where

$$W_f(x) = \prod_{p \le x} \left(1 + \frac{f(p)}{p}\right). \tag{16}$$

The upper bound follows from [8], Theorem 3.5, and

$$\sum_{n \le x} \frac{f(n)}{n} \asymp W_f(x). \tag{17}$$

The lower bound follows from (17) as in [4], Theorem A.4 (there the multiplicative function is supported on squarefree numbers, but the argument works in the general case as well). The estimate (17) is proved in [4], Theorem A.3, but unfortunately only under stronger assumptions on f (see also [1], Lemma 1, where (17) is proved under a different set of conditions which is again stronger than ours). We sketch the proof under our assumptions. Let $g(n) = f(n)/n$. For $z = x^\alpha$, with $\alpha > 0$ sufficiently small, set $g_z(m) = g(m)$ if every prime divisor $p|m$ satisfies $p \le z$ and $g_z(m) = 0$ else. Then g_z is multiplicative. With

$$F_g(x,z) = \sum_{m \le x} g_z(m), \qquad R_g(x,z) = \sum_{m > x} g_z(m)$$

it follows that

$$F_g(x,z) + R_g(x,z) = W_g^*(z) := \prod_{p \le z} \left(\sum_{\nu \ge 0} g(p^\nu) \right). \tag{18}$$

Using $\log m = \sum_{q^\nu \| m} \log q^\nu$ one obtains

$$R_g(x,z) \log x \le \sum_{m > x} g_z(m) \log m \le \sum_{n \ge 1} g_z(n) \sum_{\substack{q \le z, \nu \ge 1; q \nmid n \\ nq^\nu > x}} g_z(q^\nu) \log q^\nu.$$

By (9) and (10) the inner sum is $O(\log z)$. Note that (9) implies $\sum_{p \le x} g(p) \log p \ll \log x$. This yields $R_g(x,z) \log x \ll W_g^*(z) \log z$. Choosing the number $\alpha > 0$ sufficiently small, one finds $R_g(x,z)/W_g^*(z) \le 1/2$ for x sufficiently large. Together with (18) this yields $\frac{1}{2} W_g^*(z) \le F_g(x,z) \le F_g(x) \le W_g^*(x)$. Now (17) follows from

$$W_g^*(x) \asymp W_f(x), \qquad W_f(x) \asymp_\alpha W_f(x^\alpha). \tag{19}$$

The following lemma contains the basic counting argument for the proof of Theorem 2. It gives an upper bound of

$$A_f(x) = \#\{n \ge 1 : x f(n)/n \ge 1\} \tag{20}$$

which allows us to prove $N_f(x) \ll x$.

Lemma 4. *Let f be a non-negative multiplicative function which satisfies (9) and (10) with some $a > 1$. Additionally, assume that $f(n) \ge n^{-2}$. Then, for every $0 < \delta \le 1$ and $x \ge 3$,*

$$A_f(x) \le B \frac{x^{1+\delta}}{\log x} \sum_{n \ge 1} \left(f(n) n^{-1}\right)^{1+\delta}, \tag{21}$$

where B is a constant which depends on C_1, C_2 and a.

Proof. Let $g(n) = f(n)/n$. We start from the identity

$$\sum_{xg(n)\geq 1} \log\big(xg(n)\big) - \sum_{xg(n)\geq 1} \log g(n) = A_f(x)\log x.$$

The assumption $f(n) \geq n^{-2}$ implies $\log g(n) \geq -3\log n$. It follows that

$$A_f(x)\log x \leq \sum_{xg(n)\geq 1} \log(xg(n)) + 3\sum_{xg(n)\geq 1} \log n =: S_1 + 3S_2.$$

Using $\log u \leq u$ we obtain

$$S_1 \leq \sum_{xg(n)\geq 1} xg(n) \leq x^{1+\delta}\sum_{n\geq 1} g(n)^{1+\delta}.$$

To estimate S_2 we write

$$S_2 = \sum_{xg(n)\geq 1}\sum_{p\| n}\log p + \sum_{xg(n)\geq 1}\sum_{\substack{p^\nu \| n\\ \nu\geq 2}} \log p^\nu =: S_2' + S_2''.$$

Splitting S_2' according to the cases $f(p) \leq 1$ and $2^j < f(p) \leq 2^{j+1}$, $j \geq 0$, and using (9) we find

$$\begin{aligned} S_2' &= \sum_{\substack{m,p;\ p\nmid m\\ xg(p)g(m)\geq 1}} \log p \\ &\leq \sum_{xg(m)\geq 1}\left(\sum_{p\leq xg(m)}\log p + \sum_{j\geq 0}\sum_{p\leq 2^{j+1}xg(m)} (f(p)2^{-j})^a \log p\right) + O(1) \\ &\ll \sum_{xg(m)\geq 1} xg(m) \ll x^{1+\delta}\sum_{m\geq 1} g(m)^{1+\delta}. \end{aligned}$$

Finally, the crude estimate

$$A_f(x) = \sum_{xg(n)\geq 1} 1 \leq x^{1+\delta}\sum_{n\geq 1} g(n)^{1+\delta}$$

yields

$$S_2'' = \sum_{\substack{m,p,\nu\geq 2;\ p\nmid m\\ xg(p^\nu)g(m)\geq 1}} \log p^\nu \leq \sum_{p,\nu\geq 2} A_f(xg(p^\nu))\log p^\nu \leq C\,x^{1+\delta}\sum_{n\geq 1} g(n)^{1+\delta},$$

where

$$\begin{aligned} C &= \sum_{p,\nu\geq 2} g(p^\nu)^{1+\delta}\log p^\nu \leq \sum_{g(p^\nu)>1} g(p^\nu)^2\log p^\nu + \sum_{g(p^\nu)\leq 1} g(p^\nu)\log p^\nu \\ &\leq C_2^3 + C_2. \end{aligned} \tag{22}$$

Here we used the fact that there are at most C_2 pairs (p, ν) with $g(p^\nu) > 1$. $\square$

We use this lemma to prove

$$A_f(x) \ll \frac{x}{\log x} W_f(x). \tag{23}$$

We first observe that we may assume $f(n) \geq n^{-2}$. Otherwise set $f_1(p^\nu) = f(p^\nu)$ if $f(p^\nu) \geq p^{-2\nu}$, and $f_1(p^\nu) = p^{-2\nu}$ else. Then $f_1 \geq f$ satisfies (9)–(11). Assuming the validity of (23) for f_1, we find

$$A_f(x) \leq A_{f_1}(x) \ll \frac{x}{\log x} W_{f_1}(x) \ll \frac{x}{\log x} W_f(x).$$

Assuming $f(n) \geq n^{-2}$, Lemma 4 with $\delta = (\log x)^{-1}$ yields

$$A_f(x) \ll \frac{x}{\log x} L(\delta),$$

where $L(\delta) = \sum_{n \geq 1} g(n)^{1+\delta}$. Furthermore,

$$\log L(\delta) = \sum_p \log \left(1 + \sum_{\nu \geq 1} g(p^\nu)^{1+\delta} \right) \leq \sum_p g(p)^{1+\delta} + \sum_{p, \nu \geq 2} g(p^\nu)^{1+\delta}.$$

By (22) the second sum is bounded by a constant. Since (9) implies $g(p) \leq 1$ for $p \geq p_0$, we obtain

$$\begin{aligned} \sum_p g(p)^{1+\delta} &\leq \sum_{p \leq x} g(p) + \sum_{2^{j+1} \geq x} 2^{-j(1+\delta)} \sum_{2^j < p \leq 2^{j+1}} f(p)^{1+\delta} \log p + O(1) \\ &\leq \sum_{p \leq x} g(p) + O(1). \end{aligned}$$

This yields $\log L(\delta) \leq \sum_{p \leq x} g(p) + O(1) \leq \log W_f(x) + O(1)$ and completes the proof of (23).

Now Theorem 2 follows easily. From (15) we know that there is a constant $C > 0$ such that

$$F_f(x) \geq C \frac{x}{\log x} W_f(x) \quad (x \geq 2).$$

Write

$$N_f(x) = \#\{n \geq 1 : x f(n) \geq F_f(n)\} \leq x + N_1 + N_2,$$

where

$$\begin{aligned} N_1 &= \# \left\{ x < n \leq x^2 : x f(n) \geq C \frac{n}{\log x^2} W_f(x) \right\}, \\ N_2 &= \# \left\{ n > x^2 : x f(n) \geq C \frac{n}{\log n} \right\}. \end{aligned}$$

By (23) and (19)

$$N_1 \leq A_f \left(\frac{x \log x^2}{C W_f(x)} \right) \ll \frac{x \log x^2}{W_f(x) \log x} W_f(x^2) \ll x.$$

To bound N_2 we apply (15) to f^a. Note that (9) implies $\sum_{p \leq x} f^a(p)/p \leq C_1 \log\log x + O(1)$ via partial summation. This yields $W_{f^a}(x) \ll (\log x)^{C_1}$ and with (15) the crude bound $F_{f^a}(x) \ll x(\log x)^{C_1}$. This implies

$$N_2 \le \sum_{n>x^2} \left(\frac{x f(n) \log n}{Cn} \right)^a \ll x^a \sum_{2^{j+1} \ge x^2} 2^{-ja} j^a \sum_{2^j < n \le 2^{j+1}} f^a(n)$$
$$\ll x^a \sum_{2^{j+1} \ge x^2} 2^{-j(a-1)/2} \ll x. \quad \square$$

Acknowledgment

We thank the referee for his remarks on the paper, leading to a substantial improvement of the presentation.

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